Energetic criterion for a micro-crack of finite length initiated in orthotropic bi-material notches

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INTRODUCTION

Joints of different materials occur in practical engineering structures or in parts of electronic devices (e.g. layered composite materials, constructions with protective surface layers, thermal barriers). They enable achievement of properties which could not be attained by means of homogeneous materials. In the case of composite materials, parts of the joints often exhibit orthotropic material properties.

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INTRODUCTION

In comparison to a crack in homogeneous media, in the case of bi-material joints, the stress singularity exponent is different from 1/2 and can generally be complex. The stress is mostly characterized by more singular terms and at the same time each singular term covers combination of both normal and shear modes of loading.

In the contribution the orthotropic bi-material notch is analyzed from the perspective of generalized linear elastic fracture mechanics, i.e. the validity of small-scale yielding conditions is assumed. It is further assumed ideal adhesion at the bi-material interface.

The aim is to suggest a procedure for
• the determination of stress exponent singularity
• the evaluation of the generalized stress intensity factors
• the description of tangential stresses near the notch tip
• the evaluation of the ERR
• the determination of the crack initiation direction from an orthotropic bi-material notch
• the critical loading conditions.

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The necessary step for the stress intensity factor and crack initiation assessment is detailed knowledge of the stress distribution. Within plane elasticity of anisotropic media the Lekhnitskii-Eshelby-Stroh (LES) formalism is used. Complex potentials satisfying the equilibrium and the compatibility conditions as well as the linear stress-strain dependence.

\[ u_i = 2\Re\{A_{ij}f_j(z_j)\} \]
\[ \sigma_{2j} = 2\Re\{L_{ij}f'_j(z_j)\} \]
\[ \sigma_{1j} = -2\Re\{L_{ij}\mu_jf'_j(z_j)\} \]

\[ A = \begin{bmatrix}
    s_{11}\mu_1^2 + s_{12} & s_{11}\mu_2^2 + s_{12} \\
    s_{12}\mu_1 + s_{22}/\mu_1 & s_{12}\mu_2 + s_{22}/\mu_2
\end{bmatrix} \]
\[ L = \begin{bmatrix}
    -\mu_1 & -\mu_2 \\
    1 & 1
\end{bmatrix} \]

\[ f = H \langle z^{\delta}_* \rangle v \]
\[ \langle z^{\delta}_* \rangle = \text{diag}[z^{\delta}_1, z^{\delta}_2] \]

- \( H \) – generalized stress intensity factor (GSIF)
- \( v \) – complex eigenvector
- \( \delta \) – exponent of the stress singularity at the notch tip
- \( \mu_i \) – eigenvalues of the materials
- \( s_{ij} \) – material compliances

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EIGENVALUES OF STRESS SINGULARITY EXPONENTS AND EIGENVECTORS

Boundary and continuity condition

\[ T^I = T^II = 0 \quad \text{for} \quad \theta = \omega_1, -\omega_2 \]
\[ T^I = T^II, u^I = u^II \quad \text{for} \quad \theta = 0 \]

Homogenous algebraic equations and characteristic equation

\[ K(\delta)v = 0 \]
\[ \det(K(\delta)) = 0 \]

\[ \delta, v \] – regular solution
\[ -\delta, \tilde{v} \] – auxiliary solution

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EIGENVALUES OF STRESS SINGULARITY EXPONENTS

\[ E_L^I = 100 \text{ MPa} \]
\[ E_T^I = 50 \text{ MPa} \]
\[ \nu_L^I = \nu_T^I = 0.3 \]
\[ G_L^I = G_T^I = 30 \text{ MPa} \]
\[ E_L^{II} = 400 \text{ MPa} \]
\[ E_T^{II} = 50 \text{ MPa} \]
\[ \nu_L^{II} = \nu_T^{II} = 0.3 \]
\[ G_L^{II} = G_T^{II} = 30 \text{ MPa} \]

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In order to determine the final stress distribution around the notch, it is important to find out the value of the GSIFs from the analytical-numerical solution via the so-called \( \psi \)–integral corresponding to a concrete situation with given geometry, materials and boundary conditions. This method is an implication of Betti’s reciprocity theorem which in the absence of body forces states that the integral is path-independent.

\( u_j \) – the regular solution belongs to the exponent \( \delta \),

\( \tilde{u}_j \) – the auxiliary solution belongs to the exponent \( \tilde{\delta} = -\delta \),

\( \varphi_j \) – the stress function depending on \( \theta \).
GSIFs EVALUATION

\[ H = \frac{\psi(u^{FEM}, \tilde{u})}{\psi(u, \tilde{u})} \]

\[
\begin{align*}
E_L^I &= 50 \text{ MPa} \\
E_T^I &= 100 \text{ MPa} \\
\nu_L^I &= \nu_T^I = 0.3 \\
G_L^I &= G_T^I = 30 \text{ MPa} \\
E_L^{II} &= 400 \text{ MPa} \\
E_T^{II} &= 50 \text{ MPa} \\
\nu_L^{II} &= \nu_T^{II} = 0.3 \\
G_L^{II} &= G_T^{II} = 30 \text{ MPa}
\end{align*}
\]

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Material 1,2: $G_{ZT} = G_{ZL} = G_{TL} = 30$ GPa, $\nu_{TZ} = \nu_{ZL} = \nu_{TL} = 0.3$

Material 1: $E_{L1} = 100$ GPa, $E_{T1} = E_{Z1} = 50$ GPa
Material 2: $E_{L2} = 50$ GPa, $E_{T2} = E_{Z2} = 200$ GPa

$\delta_1 = 0.626$

$H_1 = 5.160$ MPa m$^\delta_1$

$\delta_2 = 0.927$

$H_2 = 327.436$ MPa m$^\delta_2$

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MEAN VALUE OF THE TANGENTIAL STRESS

\[
\bar{\sigma}_{\theta\theta}(\theta) = \frac{1}{d} \int_0^d \sigma_{\theta\theta}(r, \theta) dr = \frac{1}{d} \sum_{k=1}^2 \left( m_1 \varphi_1(d, \theta, \delta_k) + m_2 \varphi_2(d, \theta, \delta_k) \right)
\]

\[
\bar{\sigma}_{\theta\theta}(\theta) = H_1 F_{\theta\theta1m}(\theta) + H_2 F_{\theta\theta2m}(\theta)
\]

The stress field around a bi-material notch inherently covers combined normal and shear modes of loading. For mixed mode fields a crack may grow along the interface or at a certain angle with respect to the interface into material I or II. The MTS criterion states that the crack is initiated in the direction, where the circumferential stress at some distance from the crack tip has its maximum and reaches a critical tensile value. The local maximum of the tangential stress in the case of a bi-material orthotropic notch depends on the radial distance \( r \) from the notch tip. In order to suppress the influence of the distance \( r \), the mean value of the tangential stress is evaluated over a certain distance \( d \). The distance \( d \) has to be chosen in dependence on the mechanism of a rupture, e.g. as a dimension of a plastic zone or as a size of material grain.

\[ d = \frac{1}{\pi} \left( \frac{K_C}{\sigma_o} \right)^2 \]

\[ H_1 = 0.73 \text{ MPa}^{1-\delta_1} \]
\[ H_2 = 31.5 \text{ MPa}^{1-\delta_2} \]
\[ d = 1 \times 10^{-5} \]

\[ E_L^I = 100 \text{ MPa} \]
\[ E_T^I = 50 \text{ MPa} \]
\[ \nu_L^I = \nu_T^I = 0.3 \]
\[ G_L^I = G_T^I = 30 \text{ MPa} \]
\[ E_L^II = 400 \text{ MPa} \]
\[ E_T^II = 50 \text{ MPa} \]
\[ \nu_L^II = \nu_T^II = 0.3 \]
\[ G_L^II = G_T^II = 30 \text{ MPa} \]

\[ \theta_{\text{max}} = -76.8^\circ \]

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The direction of the applied stress is not specified directly, but it is expressed by varying ratios $\Gamma_{21} = H_2/H_1$. The angles are determined on the basis of finding the maximum of the mean value of the tangential stress. The averaging distance $d$ was taken as $1e-4$, $1e-5$ and $1e-6$. It is shown that the direction depends on $d$ especially for larger differences between $H_1$ and $H_2$. 

$\delta_1 = 0.573$

$\delta_2 = 0.941$

$E^I_L = 100$ MPa
$E^I_T = 50$ MPa
$\nu^I_L = \nu^I_T = 0.3$
$G^I_L = G^I_T = 30$ MPa
$E^{II}_L = 400$ MPa
$E^{II}_T = 50$ MPa
$\nu^{II}_L = \nu^{II}_T = 0.3$
$G^{II}_L = G^{II}_T = 30$ MPa

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The stability criterion based on the average stress calculated across a distance $d$ from the wedge tip is presented. Assumption:

- the value of the average stress corresponding to the bi-material notch is calculated for the direction received from mean value of the tangential stress,
- $K_{IC}$ is the stress intensity factor corresponding to the crack in homogenous media propagating and loading under mode I,
- consider that the ratio of the values $H_1$ and $H_2$ is constant for a given bi-material configuration and boundary conditions,
- the boundary conditions do not depend on the absolute value of the applied stress.

$$H_{1C} = \frac{2K_{IC}}{\sqrt{2\pi d} \left( F_{\theta m} (\theta_0) + \Gamma_{21} F_{\theta m} (\theta_0) \right)}$$

$$\Gamma_{21} = \frac{H_2}{H_1} = \frac{H_{2C}}{H_{1C}}$$
CHANGE OF POTENTIAL ENERGY

\[ \delta W = \psi(U^0, U^\varepsilon) \]

Asymptotic expansion for the notch before the perturbation

\[ U^0(x) = H_1 r^{\delta_1} u_1(\theta) + H_2 r^{\delta_2} u_2(\theta) + \ldots \]

Asymptotic expansion for the perturbed domain

\[ \varepsilon = a / L \]

\[ a - \text{finite length of perturbing crack} \]

\[ U^\varepsilon(x) = U^0(x) + k_1(\varepsilon)K_{1d(p)} r^{-\delta_1} u_{-1}(\theta) + k_2(\varepsilon)K_{2d(p)} r^{-\delta_2} u_{-2}(\theta) + \ldots \]

Matched asymptotic procedure is used to derive the change of potential energy for crack initiated from the notch tip.

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MATCHED ASYMPTOTIC PROCEDURE

\[ U|_{\partial \Omega^a} = \rho^{\delta} u_1(\theta) : \quad K_{1d(p)} = \frac{\Psi(V_{1h}^p, \rho^{\delta} u_1)}{\Psi(\rho^{\delta} u_{-1}, \rho^{\delta} u_1)} \]

\[ U|_{\partial \Omega^b} = \rho^{\delta} u_2(\theta) : \quad K'_{1d(p)} = \frac{\Psi(V_{2h}^p, \rho^{\delta} u_1)}{\Psi(\rho^{\delta} u_{-1}, \rho^{\delta} u_1)} \]

\[ K_{2d(p)} = \frac{\Psi(V_{1h}^p, \rho^{\delta} u_2)}{\Psi(\rho^{\delta} u_{-2}, \rho^{\delta} u_2)} \]

\[ K'_{2d(p)} = \frac{\Psi(V_{2h}^p, \rho^{\delta} u_2)}{\Psi(\rho^{\delta} u_{-2}, \rho^{\delta} u_2)} \]

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ENERGY RELEASE RATE

\[
G = - \frac{\psi(U^0, U^\varepsilon)}{2\varepsilon L} =
\]

\[
= - \frac{1}{2L} \left[ H_1^2 \varepsilon^2 \delta_1^{-1} \psi_{11} + H_1 H_2 \varepsilon^\delta_1 + \delta_2^{-1} (\psi_{21} + \psi_{12}) + H_2^2 \varepsilon^2 \delta_2^{-1} \psi_{22} \right]
\]

\[
\psi_{ij} \equiv \psi(V_i^h, q^{\delta_i u_j})
\]

\[\varepsilon = a/L - \text{the ratio between the perturbing finite crack length } a \text{ and characteristic length } L,\]

\[\psi_{ij} - \text{two-state integral calculated along the path enclosing the notch tip on the scaled-up domain } \Omega_{in}\]

\[q = r/\varepsilon - \text{scaled up polar coordinate},\]

\[u_i - \text{the basis functions of the Williams asymptotic expansion}\]

\[V_i^h - \text{the FEM solution on the scaled-up domain } \Omega_{in} \text{ corresponding to prescribed boundary conditions } u_{|\partial\Omega} = q^{\delta_i u_i}.\]
ENERGY RELEASE RATE

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\[ \delta_1 = 0.573 \]
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\[ H_1 = 0.73 \text{ MPam}^{1-\delta_1} \]
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\[ \varepsilon = 1 \times 10^{-5} \]

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Failure of the notch will occur if there is sufficient energy available to allow a finite amount of the crack growth $a$.

From the suggestion of the same mechanism of rupture and consequently the same value of ERR corresponding to threshold conditions in the case of a bi-material notch and a crack follows:

$$G_c = \frac{1}{2} \left( \mathcal{L}_{11} \mathcal{L}_{22} - \mathcal{L}_{12}^2 \right)^{-1} \mathcal{L}_{11} K_{lc}^2$$

$\mathcal{L}_{ij}$ - the components of the Barnett-Lothe tensor.
**CRACK LENGTH INCREMENT**

\[ a = \frac{G_C s_\theta^2(\theta_0)}{K(\theta_0)\sigma_{\theta\theta,c}^2} \]

- \( K(\theta_0) \) - scaling term depending on the local geometry (e.g. the angle of the notch, the inclination of the main crack, etc.) and on the direction of fracture.
- \( s_\theta(\theta_0) \) - shape function corresponding to the singular term in the Williams asymptotic expansion.
- \( \sigma_{\theta\theta,c}, G_C \) - critical values of the tangential stresses and ERR, respectively, of the penetrated material.

One possible way of the estimation of the crack length increment \( a \) offers the finite fracture mechanics. From this theory, the lower bound for the increment is given from the change of the potential energy, the upper bound for the increment follows from the critical tensions ahead of the crack tip.

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PATH INDEPENDENCY OF $\Psi$ - INTEGRAL

$$K_{1d(p)} = \frac{\Psi(V_h, \rho^{\delta_h}u_1)}{\Psi(\rho^{-\delta_h}u_{-1}, \rho^{\delta_h}u_1)}.$$
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